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Why the nature needs 1/f-noise

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Low-frequency 1/f-noise occurs at all levels of the nature organization and became an actual factor of nanotechnologies, but in essence it remains misunderstood by its investigators. Here, once again it is pointed out that such the state of affairs may be caused by uncritical application of probability theory notions to physical random phenomena, first of all the notion of "independence". It is shown that in the framework of statistical mechanics no medium could provide an inner wandering particle with quite certain value of diffusivity and mobility, thereby producing flicker fluctuations of these quantities. This is example of realization of universal 1/f-noise origin in many-particle systems: dependence of time progress of any particular relaxation or transport process on the whole system's detailed initial microstate.

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CONTENTS

I. INTRODUCTION

A. Root of the question and popular hypothesis

Here, as for many years before, the question of 1/f-noise grows in urgency, by extending and deepening together with physical experiments and new technologies and concerning almost all in the world, from cosmic phenomena down to molecular biology and nano-electronics. Nevertheless, there are no modern reviews of the question proportional to its volume and significance. Seemingly, this is so because investigators do not find an inspirational ideas and happy thoughts for that. Though, one appropriate suggestion, - to be under consideration below, - was made already in [1–4], but it had not excited a response. Somehow or other, today we see reports on more and more inventive and fine measurements of 1/f-noise, - for example, in films of metals and alloys [5] or atomic layers of graphene [6], - but as before with no unambiguous indication in its origin. Citing [6], "... despite almost a century of research, 1/f noise remains a controversial phenomenon and numerous debates continue about its origin and mechanisms".

It can be added that "debates", in the author's experience, not rarely take rather totalitarian forms. Maybe, partly by this reason, from the author's viewpoint, the present situation in general very slightly differs from what was outlined in [1] and a little later in [7]. We would like to compare it with situation in astronomy nearly three hundreds years ago before appearance of the celebrated I. Newton's work [8].

We venture such the comparison not for the sake of witticism but in view of our intention to demonstrate in the present notes that just the Newton's laws of mechanics may be the place where solution of the 1/f-noise problem is hidden. More precisely, 1/f-noise is permanent property of systems of many particles moving and interacting by these laws (in their classical or quantum formulation including fields). In order to recognize it, we have only

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to follow the Newton's advice to avoid unnecessary hypotheses ("Hypotheses non fingo" [8]).

Well, what hypotheses are thought up by physicists in respect to 1/f-noise? Let us decipher it by example of electric current noise in a conductor under fixed voltage. Presence of 1/f-noise there means that the current has no certain value, in the sense that its averaging (smoothing) over time produces an unpredictable result, - that is randomly varying from one experiment to fnother, - with a diversity which practically does not decrease, or even increases, when the averaging duration grows (since related narrowing of frequency band contributing to the diversity is almost, or with excess, compensated by growth of the noise power spectral density inside that band). So, when asking oneself a question about origin of such the phenomenon, one first of all assumes that it is in some specific fluctuation processes influencing the current, - through e.g. number of charge carriers or their mobility, - while specificity of these processes is in extremely wide variety of their time scales (memory, or life, or relaxation, or correlation times, etc.) [5, 6, 9]. Just this is main hypothesis.

B. Idea of the answer and plan of doing

Really, this hypothesis is not necessary, since the mechanics as it is in no way requires certainty of the current and, hence, some special reasons for its uncertainty. Indeed, no matter what a concrete mechanism of conductivity may be, if it is indifferent in respect to amount of charge early transported through the conductor from one side of an outer electric circuit to another and thus to past value of time-smoothed current, then later this mechanism also will be indifferent in respect to them, and on the whole it will be unable to set conditions for certainty of the current. Thus, it by itself serves as mechanism of 1/f-noise.

What is for the indifference, it is supported by the experiment conditions in themselves which state that fluctuations of (time-smoothed) current do not meet a back reaction of outer circuit instead passively swallowing them.

In this reasoning, there is no collections of large characteristic times, instead a single time only is present, - usually small in practice, - which indicates ending of memory of conductance mechanism. If, for instance, it is less than several hours, then transfer of charge carriers, - with their collisions, scatterings, reflections, etc., - now, at present time interval, is passing indifferently to what amount of charge was transported yesterday, even if an experimental device was not switched off before going to bed. Correspondingly, at frequencies lower than inverse day one can find a 1/f-noise.

Analogously, if somebody possesses unlimited possibilities of profits and expenses and does not keep count of them, then he himself could not know how much his expenditures may be on average over time, and it can be expected that they will be distributed in time like 1/f-noise.

If, returning to the conductor, we short out it, then the current's 1/f-noise disappears along with directed current. But irregular charge displacements in opposite directions do continue, at that again indifferently to their past amount, and thus to their time-average intensity. The latter therefore is not aimed at a certain value, which results in 1/f-fluctuations of intensity (power spectral density) of thermodynamically equilibrium "white" (thermal) current noise. They are connected to the 1/f-noise in non-equilibrium current-carrying conductor by means of the "generalized fluctuation-dissipation relations" [1, 2, 10–12].

If one measures equilibrium thermal noise of potential difference between sides of opened conductor, - e.g. electric junction, - then 1/f-fluctuations of intensity of this noise can be found too. They say that sum of numbers (per unit time) of random charge carrier transitions from one side to another and backwards is not tracked and regulated by the system, in contrast to residial of that numbers [10]. At that, characteristic time constant of the system (equivalent RC-circuit) determines upper time scale for fluctuations of the residial and lower one for fluctuations in the sum (while their upper time scale does not exist, since they do not change system's macrostate).

The aforesaid can be easy extended, - under nonprincipal substitutions of particular terms and meanings, - to other manifestations of 1/f-noise in the nature. A lot of various examples was exposed in [1, 2, 4, 10, 11, 13, 21]. Our demonstration below will be realized in terms of equilibrium "molecular Brownian motion" [3, 7, 10, 12, 14–20, 22].

In a maximally simple way we shall show that assumption that Brownian particle obeys a certain diffusivity (rate, or coefficient, of diffusion) is incompatible with exact equations of statistical mechanics, that is with mechanical dynamical background of Brownian motion. Thus, mechanics inevitably generates 1/f-, or "flicker", fluctuations of diffusivity and mobility of the particle.

Then we shall consider quantitative characteristics of this 1/f-noise and, finally, present its explanation in the language of theory of deterministic chaos in many-particle systems.

II. PHENOMENOLOGY OF BROWNIAN MOTION

A. Formulation of problem

Let us imagine a small "Brownian particle" in a three-dimensional statistically uniform isotropic and thermodynamically equilibrium medium. Very small particle of dust or flower pollen, - whose motion in liquid for the first time was observed through microscope in [23, 24] and in the beginning of next century theoretically analysed in [25–27], - are suitable objects. But it will be better to

take in mind some "nano-particle" or even merely separate atom or molecule in liquid or gas [28]. In principle, we may speak even about free charge carrier or point defect in a solid, but confine ourselves by a particle which quite definitely is subject to the classical variant of mechanics.

Let $R(t)$ and $V(t) = dR(t)/dt$ denote vectors of centre-of-mass coordinate and velocity of our Brownian particle (BP) at given time instant, while R and V their possible values. We can think that initially at time $t = 0$ BP was placed at definitely known space point. Where namely, is of no importance, because of thermodynamical equivalence of any BP's positions. Therefore it is convenient to choose the coordinate origin: $R(0) = 0$. Then later instant current position of BP, $R(t)$, will be coinciding with vector of its total displacement, or path, during all previous observation time.

Now, let us ask ourselves what is BP's "diffusion law", i.e. probability distribution of BP's path. Density of this distribution will be designated by $W(t, R)$. It can be represented by expression

$$W(t, R) = \langle\, \delta(R - R(t))\, \rangle\,, \tag{1}$$

where the Dirac delta-function figures, $R(t)$ is thought as result of all the previous interaction vetween BP and the medium, and the angle brackets designate averaging over the equilibrium (Gibbs [29]) statistical ensemble of initial states of the medium and initial values of BP's velocity.

Undoubtedly, a plot (relief) of $W(t, R)$ as function of R looks like a "bell" extending and lowering with time. We are interested in what shapes of this bell may be formed in reality.

B. Conditional averaging and continuity equation

In fact, (1) is mere identity, but its time differentiation immediately brings us a food for thought. From it we have

$$\frac{\partial W(t, R)}{\partial t} = -\nabla \cdot \langle V(t)\, \delta(R - R(t))\rangle$$

($\cdot$ will denote scalar product of vectors). By attracting mathematical tools of the probability theory [33], this equality can be rewritten as

$$\frac{\partial W(t, R)}{\partial t} = -\nabla \cdot \overline{V}(t, R)\, W(t, R)\,, \tag{2}$$

where $\overline{V}(t, R) = \langle V(t)\rangle_R$ is conditional average value of BP's instant velocity determined under condition that its current position, and thus its previous path, is known (measured) to be equal to $R(t) = R$. Generally the operation of conditional averaging $\langle \dots \rangle_R$ is defined by formula

$$\langle \dots \rangle_R \equiv \langle \dots\, \delta(R(t) - R)\rangle / \langle \delta(R(t) - R)\rangle\,.$$

Obviously, (2) is "continuity equation" for the probability density $W(t, R)$, and the "field of velocity of probability flow", $\overline{V}(t, R)$, containes important information about solutions to this equation. Therefore, first of all let us consider possible constrution of the vector-function $\overline{V}(t, R)$.

C. Conditional average velocity of Brownian particle

We shall keep in mind that the duration t of our observations of BP is much longer than characteristic relaxation time τ of (fluctuations of) BP's velocity.

Then, firstly, apply heuristic reasonings as follow. On one hand, by the condition $R(t) = R$, average value of BP's velocity in the past, at time of its preceeding observation, appears equal to R/t. On the other hand, as far as BP makes a random walk and $t \gg \tau$, the same condition $R(t) = R$ tells us almost nothing about BP's velocity in the future, so its average value in equal next time interval can be expected to be zero. Hence, since the average under question, $\overline{V}(t, R)$, relates to the present time instant "in the middle between past and future", it seems likely that it is equal to half-sum of the mentioned quantities:

$$\overline{V}(t, R) = \frac{R}{2t}\,. \tag{3}$$

We can confirm this conclusion in a more formally rigorous way, basing on the main distinctive statistical property of Brownian motion [26]:

$$\langle R^2(t)\rangle = \int R^2\, W(t, R)\, dR = 6Dt \tag{4}$$

at $t \gg \tau$, i.e. ensemble average of squared BP's displacement grows proportionally to observation time. It is sufficient to notice that the continuity equation implies

$$\frac{\partial}{\partial t} \int R^2\, W\, dR = 2 \int R \cdot \overline{V}\, W\, dR$$

and that this requirement is naturally satisfied together with (4) (with taking account of parallelity $\overline{V} \parallel R$) when equality (3) is valid.

By way, notice that BP's diffusivity, or diffusion coefficient, D and relaxation time τ always can be connected via relation

$$D = V_0^2\, \tau \equiv \frac{T}{M}\, \tau\,,$$

where T is temperature of the medium, M is mass of BP, and $V_0 = \sqrt{T/M}$ its chracteristic thermal velocity.

D. General form of probabilistic law of diffusion and uncertainty of its coefficient

After inserting function (3) into (2), one comes to partial differential equation

$$2t\,\frac{\partial W}{\partial t} = -3\,W - R\cdot\nabla W\,, \qquad (5)$$

which clearly indicates scale-invariant character of its solutions. We are interested in isotropic (spherically symmetric) solutions looking as

$$W(t,R) = (2Dt)^{-3/2}\,\Psi(R^2/2Dt) \qquad (6)$$

with some dimensionless function $\Psi(z)$ of dimensionless argument $z = R^2/2Dt$. In our context it, since representing probability density, anyway should be non-negative and satisfying normalization condition $\int W\,dR = 1$ in company with equality (4), which surely can be done. Then (6) is most general law of *diffusional* random walk, when typical BP's displacements are proportional to square root of observation time: $R^2(t) \propto t$.

In particular, taking $\Psi(z) = (2\pi)^{-3/2}\,\exp(-z/2)$, one obtains the commonly known Gaussian diffusion law,

$$W = W_D(t,R) \equiv (4\pi Dt)^{-3/2}\exp(-R^2/4Dt)\,. \qquad (7)$$

The corresponding walk is much pleasant for users since in rough enough, in comparison with τ, time scale its successive increments are mutually statistically independent. Owing to this, the only parameter of such random walk, - its diffusion coefficient, or diffusivity, D - can be unambiguously determined from observations of any its particular realization, by means of long enough time averaging.

However, similar observations and time averaging of non-Gaussian random walk, obeying some of general type distributions (6), will produce every time different values of diffusivity [1–4, 7, 10]. Indeed, their coincidence, that is convergence of all results of time averaging to one and the same value, would be impossible without statistical independence of increments (at least mutually far time-distanced ones) which in turn would mean, in accordance with respective limit theorem of the probability theory (the "law of large numbers"), that at $t \gg \tau$ probability distribution of total path tends to the Gaussian ("normal") one [31].

This becomes quite obvious if distribution (6) is represented by linear combination of Gaussian "bells":

$$W(t,R) = \int_0^\infty W_\Delta(t,R)\,U\left(\frac{\Delta}{D},\xi\right)\frac{d\Delta}{D}\,.$$

Such expansions naturally arise in the microscopic theory [12, 14–16]. Correspondingly, in place of $\Psi(z)$ in (6) we can write

$$\Psi(z,\xi) = \int_0^\infty \frac{\exp(-z/2\zeta)}{(2\pi\zeta)^{3/2}}\,U(\zeta,\xi)\,d\zeta\,. \qquad (8)$$

Function $U(\zeta,\xi)$ here plays role of probability distribution of $\zeta = \Delta/D$, i.e. random diffusivity of BP Δ expressed in units of its mean diffusivity D. The latter is formally defined by equality (4), while practically one may try to determine it with the help of averaging over many experiments or many copies of BP.

The additional argument ξ in this expansion, - if introduced e.g. as $\xi \equiv \tau/t$ under convention $\Psi(z,0) = \Psi(z)$, - allows to take into account violation of ideal scale invariance of random walk at $\xi \neq 0$. First of all, far on "tails" of diffusion law, where $R^2 \gtrsim V_0^2t^2$, that is $z \gtrsim 1/\xi$. There rate of diffusion achieves values of rate of free flight, $\Delta \sim V_0^2 t = D/\xi$.

Clearly, a correction of tails of diffusion law may strongly influence its higher-order statistical moments and cumulants, even in spite of $\xi \ll 1$. At that, nevertheless, shaping of $W(t,R)$'s bell in the main stays almost unchanged. Accordingly, a change of the function $\overline{V}(t,R)$, - required by equation (2) and condition (4), - is as small as ξ is, so that the expression (3) remains right.

Notice that the very possibility of long-term violation of scale invariance automatically presumes non-Gaussianity of diffusion law, since Gaussian statistics merely gives no place for it (since it would contradict the condition (4)). Already this fact gives evidence that Gaussian law is not completely adequate reflection of reality, although in mind of scientists it is firmly associated with diffusion of physical particles. At the same time neither general reasonings leading to (3) and (5) nor equation (5) by itself in no way dictate the special Gaussian choice. Therefore, it is desirable to discuss other possibilities and search for criteria of choice among them in the framework of statistical mechanics.

III. MICROSCOPIC APPROACH

A. Newton equation and Liouville equation

Further, let us go from kinematics of Brownian motion to its dynamics and directly consider BP's interaction with medium using methods of statistical mechanics. With this purpose we can take for our system quite usual simple Hamiltonian

$$H = \frac{P^2}{2M} + \Phi(R,\Gamma) + H_{th}(\Gamma)\,, \qquad (9)$$

where $P = MV$ is BP's momentum, Γ is full set of (canonical) variables of the medium, $\Phi(R,\Gamma)$ is energy of BP-medium interaction, and $H_{th}(\Gamma)$ is Hamiltonian of medium in itself (or, in other words, that of "thermostat"). If BP possesses internal degrees of freedom, then their variables will be thought included into the set Γ, thus being formally treated as a constituent of medium.

Let $\mathcal{D} = \mathcal{D}(t,R,P,\Gamma)$ designate density of full probability distribution of microstates of our system. Its evolution is described by the formally exact Liouville equation

[29, 34]. Here we can display it partly, writing out its terms only directly concerning BP:

$$\frac{\partial \mathcal{D}}{\partial t} = -V \cdot \nabla \mathcal{D} - F(R,\Gamma) \cdot \nabla_P \mathcal{D} + \dots . \tag{10}$$

Here $F(R,\Gamma) = -\nabla \Phi(R,\Gamma)$ is force acting onto BP because of its interaction with medium, and the dots surrogate terms with Γ derivatives.

Considering probability distribution of displacement (coordinate) of BP,

$$W(t,R) = \iint \mathcal{D}(t,R,P,\Gamma)\, d\Gamma\, dP \,,$$

from equation (10) after its integration over Γ and P one comes, of course, to the continuity equation (2). The same integration after multiplying (10) by V produces additional equation

$$\frac{\partial}{\partial t} \overline{V} W = -\nabla \cdot \overline{V \circ V}\, W + M^{-1} \overline{F}\, W \,. \tag{11}$$

Here and below the symbol $\circ$ denotes tensor product of vectors, while the over-line means, as before, conditional averages under given $R(t) = R$. Namely, in the first term on the left

$$\overline{V \circ V}(t,R) = \langle V(t) \circ V(t) \rangle_R = \frac{\iint V \circ V\, \mathcal{D}\, d\Gamma\, dP}{W}$$

and in second term there

$$\overline{F}(t,R) = \langle F(R(t),\Gamma(t)) \rangle_R = \frac{\iint F(R,\Gamma)\, \mathcal{D}\, d\Gamma\, dP}{W} \,.$$

Equation (11) describes momentum exchange between BP and medium. In essence, - as it can be easy verified, - this is merely the Newton equation $M\, dV/dt = F$ after its conditional averaging:

$$\langle\, M\, dV(t)/dt - F(R(t),\Gamma(t))\, \rangle_R = 0 \,.$$

We shall transform it into relation between functions $\overline{F}(t,R)$ and $W(t,R)$ which is able to help selection of acceptable diffusion laws without more deepening into the Liouville equation.

B. Equation of friction of Brownian particle

Replacing derivative $\partial W/\partial t$ in the equation (11) with right-hand side of (2), after simple manipulations one comes to equivalent exact equation

$$\frac{d\overline{V}}{dt} + \frac{\nabla \cdot \overline{\overline{V \circ V}}\, W}{W} = \frac{\overline{F}}{M} \,, \tag{12}$$

with "material derivative" of BP's average velocity,

$$\frac{d\overline{V}}{dt} = \frac{\partial \overline{V}}{\partial t} + (\overline{V} \cdot \nabla)\, \overline{V} \,,$$

and the double over-line marking tensor (matrix) of conditional quadratic cumulants (second-order cumulants) of velocity:

$$\overline{\overline{V \circ V}} \equiv \overline{V \circ V} - \overline{V} \circ \overline{V} \,.$$

Next, at first let us consider the latter object.

Since we are speaking about thermodynamically equilibrium Brownian motion, we can state that the conditional cumulants' matrix $\overline{\overline{V \circ V}}(t,R)$ at $t \gg \tau$ coincides with matrix of unconditional equilibrium quadratic statistical moments of velocity, $\langle V(t) \circ V(t) \rangle$, that is reduces to scalar number $V_0^2 = T/M$ regardless of R. Indeed, if $t \gg \tau$, then at any R the condition $R(t) = R$ fixes BP's position occurred after many random steps and cycles of momentum and energy exchange between BP and medium under detail balance in this process. Therefore, the value (variance) of corresponding thermal randomness of BP's velocity is not affected by this condition (otherwise, thermal kinetic energy of BP, on average equal to $M\overline{\overline{V \cdot V}}/2$, would be dependent on where BP is found).

The said can be confirmed by direct calculation of the matrix $\overline{\overline{V \circ V}}(t,R)$ in case of Gaussian random walk subject to distribution (7), which yields

$$\overline{\overline{V \circ V}}(t,R) = V_0^2\, (1 - \xi/2) \to V_0^2 \tag{13}$$

at $\xi \equiv \tau/t \to 0$. This result is valid also in case of non-Gaussian walk obeying (6) and (8), since its difference from Gaussian one is contained in its higher-order cumulants.

Then, let us compare two terms on left side of (12). For the first of them insertion of expression (3) gives

$$\frac{d\overline{V}}{dt} = -\frac{R}{4t^2} = -\frac{\xi}{2}\, \frac{T}{M}\, \frac{R}{2Dt} \,.$$

For the second term after insertion of (13) and (6) we have

$$\left(1 - \frac{\xi}{2}\right) \frac{T}{M}\, \frac{2\, d \ln \Psi(z,\xi)}{dz}\, \frac{R}{2Dt} \sim -\frac{T}{M}\, \frac{R}{2Dt}$$

with same shortened notation $z = R^2/2Dt$ as before. Right-hand expression here corresponds to the Gaussian diffusion law, for which $d \ln \Psi/dz = -1/2$, but by order of magnitude it is true in general case too, at least at $z \ll 1/\xi$. It shows that the first term, being approximately $2t/\tau$ times smaller than the second, is negligibly small in the limit $\xi \to 0$.

Hence, consideration of long enough time intervals leads us from (12) to shortened relation

$$-\left(\frac{T}{D} \left[-\frac{2\, d \ln \Psi(z,\xi)}{dz} \right] \right) \frac{R}{2t} = \overline{F} \,. \tag{14}$$

It resembles an equation of viscous friction, with $R/2t = \overline{V}$ in the role of velocity of a body moving through fluid and the round brackets in the role friction coefficient.

One more simplification can be obtained by neglecting, under mentioned limit, violation of scale invariance and treating $\Psi(z,\xi)$ as a function of single argument $\Psi(z)$. In the next paragraphs we firstly proceed just so.

But before that let us once again glance at the vanishing first left term of(12). If writing its contribution to the mean force as

$$M\,\frac{d\overline{V}}{dt} = -\,\nabla\,\frac{M\overline{V}^2}{2}\,,$$

one can say that this is force of reaction of the medium to addition $M\overline{V}^2/2$ to energy of BP, and system as the whole, introduced by the very measurement of BP's path, and therefore this force is not sensible to shape of diffusion law and thus to concrete peculiarities of medium. There is evident analogy with perturbing effects of measurements in quantum mechanics.

In opposite, the remaining, in the large-time limit, part of the force, (14), is determined solely by shape of probability distribution of equilibrium Brownian displacement ("diffusion law"). Consequently, this force characterizes inherent, - unperturbed by observations, - BP-medium interaction. In particular, it shows characteristic levels of the interaction forces and energies necessary for realization of one or another concrete diffusion law. Now, examine in this respect the Gaussian law (7) and make sure that it is unrealistic.

C. Paradox of Brownian motion: Gaussian statistics for it is beyond strength of its mechanics

For Gaussian diffusion law, the square bracket in the "friction equation" (14) turns to unit, and the equation becomes linear:

$$\overline{F} \,\Rightarrow\, -\frac{T}{D}\,\frac{R}{2t} = -\frac{R}{|R|}\,\sqrt{z}\,\frac{T}{\sqrt{2Dt}}\,. \qquad (15)$$

At that, the "friction coefficient" in front of $R/2t = \overline{V}$ connects to the diffusivity via relation similar to the widely known "Einstein relation" [26, 28]. Such likeness, however, is not a plus but minus of equality (15).

The matter is as follows. Friction force in the true Einstein relation represents medium resistance against directed motion of a particle. When this particle displaces by distance R, the corresponding force makes work

$$\sim |R\cdot\overline{F}| \sim \left(\frac{T}{D}\,\frac{R}{t}\right)\cdot R\,\sim\, zT\,,$$

thus producing a heat (recall that $z = R^2/2Dt$). This quantity, - like the force itself, - in principle can be arbitrary large under proper initial value of the particle's kinetic energy.

This is clear. But it is strange thing that equality (15) offers the same, also unbounded, characteristic values of force and work. Such a picture categorically contradicts to sense.

Really, - repeating the aforesaid, - in our case the force what figures in (15) represents medium reaction to particle's displacement achieved along random trajectory of thermal motion, when initial energy value knowingly is only $\sim T$. At that, the medium creates obstacles to inertial free flight of BP but in no way to its unrestricted moving off from beginning of its path. In opposite, the moving off proceeds due to medium's own free will and at the expense of its own equilibrium fluctuations.

Therefore in reality, in contrast to (15), the average force (14), as a function of passed path R, can not be arbitrary large, instead staying always and everywhere bounded. This is required by such factual inherent property of Brownian motion as translational invariance, that is indifference of the system in respect to irretrievable departures of BP anywhere. Moreover, on this ground it is reasonable to expect that at large $|R|$ the returning force vanishes at all.

Thus, we have to conclude that the Gaussian law is inadequate to physical origin of Brownian motion.

Inevitability of this conclusion catches eye when noticing that if equality (15) was true then it would mean that medium returns BP to start of its path with force proportional to the path, $\overline{F} \propto -R$, i.e. like ideal spring with potential energy $zT/2 \propto R^2$. From physical point of view this looks absurdly, since any far BP's going away is permitted just because it does not change thermodynamical state of the system.

Our conclusion can be denominated as paradoxical, if recollecting that Gaussian diffusion law many times issued from pen of theoreticians in various physical contexts and occupies important place in idealized world of "mathematical physics". But the paradox resolves in very simple way: the Gaussian statistics always had appeared as consequence of clear or implicit hypotheses (or postulates) about "independences" of random events or values. What is for us, we have managed without such hypotheses and thus showed their fallacy in application to Brownian motion.

In the past, we too were not connected with them and arrived to the same paradoxical conclusion, in the framework of both phenomenological statistical analysis of diffusion and transport processes [1–4, 10] and analysis based on the full hierarchy of Bogolyubov-Born-Green-Kirkwood-Yvon (BBGKY) equations [7, 10, 14, 16, 17, 22], as well as on the base of exact "generalized fluctuation-dissipation relations" (FDR) or "dynamical virial relations" [12, 15, 16, 20], and by other methods [10, 11, 21], including that for quantum systems [13, 19, 21].

In Section IV we shall again touch on "paradox of independence". And now, in next paragraph, consider examples of physically correct diffusion law as an alternative of Gaussian one.

D. Thermodynamics of Brownian motion and statistics of large deviations

From the left expression in (14) it is clear that the boundedness of the force $\overline{F}$ in general implies relation

$$|\overline{F}(t,R)| \,\leq\, \overline{F}_{max}(t) \,\sim\, \frac{T}{\sqrt{2Dt}}\,, \qquad (16)$$

whose right-hand part can be easy guessed for reasons of dimensionality. Of course, the symbol $\sim$ here hides some dimensionless coefficient which reflects particular distinctions of the system and construction of the function $-\ln\Psi(z)$.

Comparison between (16) and (15) shows that in the region of "tails" of diffusion law, at $z \gtrsim 1$, the Gaussian law requires to exaggerate the real force value at least $\sim \sqrt{z}$ times, thus thoroughly falsely describing (strongly underestimating) probabilities of large displacements of BP with $z \gg 1$.

In reality, according to (16), function $-\ln\Psi(z)$ grows not faster than $\propto \sqrt{z}$, so that $(-\ln\Psi(z))/\sqrt{z} < \infty$, and consequently decrease of $W(t,R)$ at large $|R| \to \infty$ is always sub-exponential (anyway, not faster than simple exponential, not speaking about "Gaussian").

The difference of reality from "Gaussian ideal" becomes aggravated when not the force itself only is bounded but also value of characteristic energy (work) conjugated with this force:

$$A(z) \equiv |R||\overline{F}|/2 \,\leq\, A_{max} = A(\infty) \,\sim\, T \qquad (17)$$

(with the same remark about $\sim$). It is natural expectation in view of that at result of any walk (any path R) the medium takes from BP not more energy than BP had been able to take from medium before.

Boundedness of $A(z)$ implies that of the force, moreover, implies that under increase of $|R|$ the force passes through a maximum and then decreases down to zero, approximately as $|\overline{F}| \,\approx\, 2A_{max}/|R| \,\propto\, T/|R|$. This asymptotic again is prompted already by dimensionality of quantities we give to disposal of statistical thermodynamics.

As the consequence, following equality (14), the tails of diffusion law and thus probabilities of large deviations ($z \gg 1$) from typical behavior ($z \sim 1$) decrease under growth of $|R|$ even much slower than in mere sub-exponential fashion generally dictated by inequality (16). Now they decrease in a power-law fashion:

$$\Psi(z) \,\propto\, z^{-A_{max}/T} \qquad (z \to \infty)\,.$$

It is seen after scalar multiplication of (14) by R, then solving so obtained differential equation, which yields

$$\Psi(z) = \Psi(0)\,\exp\left[-\int_0^z \frac{A(z)}{Tz}\,dz\right],$$

and finally applying inequality (17).

It must be underlined, besides, that boundedness of the force declared by (16) also logically implies vanishing of the force at infinity (excluding border case only when $F_{max}(t) = |F(t,\infty)|$), so that the medium's "spring" resists to small "stretching" only and always loses elasticity at large stretching.

The appropriate example of diffusion law satisfying (17), that is possessing power-law tails, is presented by

$$\Psi(z) = \frac{(3/2+\eta)!}{(2\pi\eta)^{3/2}\,\eta!}\,\left(1+\frac{z}{2\eta}\right)^{-5/2-\eta} \qquad (18)$$

with free parameter $\eta > 0$ (factorial $x!$ is standard synonym of the gamma-function $\Gamma(x+1)$). At that, obviously, $A_{max} = (5/2+\eta)\,T$. The condition $\eta > 0$ is necessary for finiteness of the mean diffusivity in (4).

Such distribution, with $\eta = 1$, for the first time was obtained in [14] from consideration of Brownian motion ("self-diffusion" [7]) of test, or "marked", atom of a gas. Similar distribution was found for molecular Brownian motion in a liquid [15, 16]). Though, strictly speaking, this is approximation of formally more exact but more complicated expressions taking into account, among other factors, violation of the scale invariance.

Formula (18) turned out to be a reasonable approximations also for BP whose mass M differs from mass m of medium (gas) atoms. At that, various mathematical approaches [17, 20, 22] to the BBGKY equations lead to identical estimate of the parameter η as a function of mass ratio, $\eta = M/m$.

Hence, investigation of complete (infinitely-many-dimensional) Liouville equation qualitatively justifiers results of our semi-heuristic analysis of initial terms of this equation.

We may further lower formal rigor and try to visually "by fingers" interpret mathematical connections between statistics of Brownian motion and its microscopic mechanism. For instance, namely, let Π be internal pressure of the medium (gas) and the quantity $A_{max} = A(\infty)$ be identified with $3T/2 + \Pi\Omega$, where Ω is gas volume forced out by far walking BP from its vicinity, and $\Pi\Omega$ is related forcing work. In essence Ω represents deficiency of BP's collisions with gas atoms facilitating its far going away. At that, since position of center of mass of the system stays fixed, an effective decrease of gas mass near BP, $m\Omega n$, - with n being mean concentration of gas atoms, - just compensate local mass excess $M+m$ accompanying current BP-atom collision. From here we have $\Omega = (M/m+1)/n$ and, taking $\Pi/n = T$ for not too dense gas, $A_{max} = (5/2 + M/m)\,T$.

Such reasonings, of course, by themselves are rather unsafe, but they can be supported by exact results on pair many-particle non-equilibrium statistical correlations [12, 15, 16, 20]. In particular, that is a theorem stating that short-range character of one-time spatial pair BP-atom correlation (boundedness of "correlation volume" Ω) implies long-range ("long-living") behavior of many-time self-correlations in BP's motion and thus in-

validity of Gaussian diffusion law (and, reciprocally, validity of the latter requires non-locality of BP-gas correlations in space) [15, 16].

E. Uncertainty and flicker fluctuations of diffusivity

The expansion (8) of non-Gaussian diffusion law (18) over Gaussian ones yields for related probability distribution of $\zeta = \Delta/D$ expression

$$U(\zeta,0) = \frac{1}{\eta!\,\zeta}\left(\frac{\eta}{\zeta}\right)^{\eta+1}\exp\left(-\frac{\eta}{\zeta}\right). \qquad (19)$$

According to the FDR [1, 4, 12] this distribution transmits onto BP's mobility (at least "low-field" one) and therefore can be observed in measurements of diversity of "time-of-flight" (time of drift) values of Brownian particles under influence of external force (for example, injected electrons or holes in semiconductors) [18].

Effects of non-Gaussian statistics were observed also directly in equilibrium, by measuring "fourth cumulants" (irreducible fourth-order correlations) of electric current or voltage noise [1]. At that, low-frequency, at frequencies $f \sim 1/t$, fluctuations of power spectral density of thermal white noise were under investigation, i.e. in essence uncertainty and fluctuations of rate of charge transfer and rates (coefficients) of diffusion of charge carriers.

In our example (18)-(19), for $\eta > 1$, it is not hard to obtain

$$\frac{\langle (R^2)^2 \rangle}{\langle R^2 \rangle^2} - 3 = 3\left(\frac{\langle \Delta^2 \rangle}{\langle \Delta \rangle^2} - 1\right) = \frac{3}{\eta - 1}$$

with $\langle\Delta\rangle = D$ and $\langle R^2\rangle = 6Dt$. This formula shows not only degree of uncertainty of diffusion rate but also defects of approximation of pure scale invariance: divergence of variance of diffusion rate at $\eta \leq 1$ and pure independence of the variance on duration of observations. The latter makes the rate fluctuations effectively quasi-static, with spectrum (power spectral density) $S_D(f) \propto D^2\,\delta(f)$ concentrated at zero frequency. This is usual result of a simplest (though non-trivial) approach to 1/f-noise from microscopic theory [13, 21].

Undoubtedly, in a more precise theory beyond ideal scale invariance [14, 17] the slow tails of diffusion law are somehow "cut off", at least at $R^2 \gtrsim V_0^2 t^2$ ($\Delta \gtrsim D/\xi \sim V_0^2 t$ in (8)), so that the Δ's variance, as well as all the higher-order statistical moments of R and Δ, definitely are finite and hardly exceed values corresponding to free BP's flight: $\langle (R^2)^k\rangle \lesssim (2k+1)!!\,(V_0^2 t^2)^k$ and $\langle \Delta^k\rangle \lesssim (2k+1)!!\,(V_0^2 t/2)^k$. What is for the delta-function $\delta(f)$, it in a definite way "spreads", with keeping dimensionality and singularity at zero, into $\sim 1/f$, where $\sim$ replaces some function of $\ln(\tau f)$.

The first of these corrections is easy describable by replacing $U(\zeta,0)$, - for instance, in (19), - by approximate expression $U(\zeta,\xi) \approx U(\zeta,0)\,\Xi(\zeta\xi)$, in which $\Xi(0) = 1$ and $\Xi(\cdot)$ in sufficiently fast way tends to zero at infinity. Then instead of (18) one obtains $\Psi(z,\xi) \approx \Psi(z)\,\Theta(z\xi)$, where scale-invariant factor $\Psi(z)$ is the same as before, - for instance, in (18), - and also $\Theta(0) = 1$ and $\Theta(\cdot)$ fast decreases to zero at infinity, thus cutting off the $\Psi(z)$'s tail. As the result, the quadratic cumulant of Δ becomes finite even at $\eta \leq 1$. At once it acquires time dependence, so that the fourth cumulant of BP's displacement increases with time $\propto t^{3-\eta}$, if $\eta < 1$, and $\propto t^2 \ln(t/\tau)$ at $\eta = 1$. Correspondingly, the quasi-static spectrum $\propto \delta(f)$ transforms to "flicker" spectrum,

$$S_D(f) \sim \frac{D^2}{\pi f}\left[\frac{1}{\tau f}\right]^{1-\eta}, \qquad (20)$$

at $\tau f \ll 1$, in particular, to $\propto 1/f$, when $\eta = 1$.

However, at $\eta > 1$ such correction is insufficient for full "spreading" of frequency delta-function, which says that scale invariance violation in this case has some another or more complex character. A notion of how else it may look we can obtain, for instance, if consider [1–4, 10, 32] diffusion law possessing property of infinite divisibility in the sense of the probability theory [31] but asymptotically only for $\xi = \tau/t \to 0$, since no real transport process can be physically divided into infinitely small independent pieces. The respective kernel in the expansion (8) simplistically is describable by formula

$$U(\zeta,\xi) \approx \frac{\alpha(\xi)\,\exp_+[-(\zeta - \zeta_0(\xi))/c]}{[\,\zeta - \zeta_0(\xi) + \alpha(\xi)]^2}, \qquad (21)$$

where $\alpha(\xi) = 1/\ln(1/\xi) = [\ln(t/\tau)]^{-1}$, function $\exp_+(x) = \exp(x)$ at $x > 0$ and $\exp_+(x) = 0$ at $x < 0$, function $\zeta_0(\xi)$ is determined by condition (4), i.e. $\int \zeta\, U(\zeta,\xi)\, d\zeta = 1$, while $c = r_0^2/D\tau_0$ with r_0 and τ_0 being minimal space and time scales down to which the "infinite divisibility" of random walk is physically meaningful ((21) presumes for simplicity that the constant c is not too small, $c \gg \alpha(\xi)$). As it is seen from here, at $\xi \to 0$ expression (21) turns to $U(\zeta,0) = \delta(\zeta - 1)$. Thus, the scale-invariant "seed" of such the diffusion law is purely Gaussian, which motivates to name it "quasi-Gaussian" [10]. In [32] it was considered in detail, including its generalizations and comparison with experiments [18].

For tails of the quasi-Gaussian law at $z \gg 1$ from (21) and (8) one can find

$$\frac{\Psi(z,\xi)}{\Psi(0,\xi)} \sim \alpha(\xi)\,\frac{2c\sqrt{\pi}}{z^{3/2}}\exp\left(-\sqrt{\frac{2z}{c}}\right),$$

that is tails satisfy the boundedness requirement (16), although lie on boundary of set of diffusion laws permitted by (16). And for spectrum of flicker fluctuations of diffusion rate, - or, generally, rate of a transport process, - the kernel (21) yields

$$S_D(f) \approx \frac{D^2 c}{f}\left[\ln\frac{1}{\tau f}\right]^{\gamma}\,, \qquad (22)$$

where $\gamma = -2$.

Spectrum (22) results mainly from difference between most probable and average (ensemble-averaged) values of the rate, namely, $\zeta_0(\xi)$ and 1 in (21) in relative units. More precisely, (22) reflects logarithmically slow decay of this difference with observation time: $1 - \zeta_0(\xi) \approx \alpha(\xi)\,\ln(c/\alpha(\xi))$.

The kernel (19) by its shape is quite similar to (21) (both consist of more or less sharp "wall" on the left and comparatively gentle slope on the right), but analogous difference in (19) is fixed. Imparting a time dependence to it may be one more, parallel, scenario of spreading of spectrum $\propto \delta(f)$ under improved analytical approximations of solutions to the BBGKY equations. From our point of view, this is practically important problem of of statistical mechanics.

Though, even presently available approximations of microscopic theory are able to realistic quantitative estimates of 1/f-noise amplitude. Estimates obtained in [7] and in [14, 18] in different approximations ((22) with $\gamma = 1$ and (20) with $\eta = 1$ or (22) with $\gamma = 0$, respectively), although differing one from another by factor $\ln[1/(\tau f)]$, nevertheless, both are in satisfactory agreement with experimental data on liquids and gases [1, 18], with taking into account diversity of these data.

On the other hand, the scheme of quasi-Gaussian random walk rather well predicts or explains level of electric 1/f-noise in various systems [1–3, 10, 18]. Since transported physical quantity there is charge instead of mass (in view of smallness of mass of usual charge carriers), and interactions of walking charges with medium is essentially long-range, it is not surprising that transport statistics there is non-Gaussian in essentially other manner than in case of molecular Brownian motion. Analysis of relation of this statistics to a quantum many-particle Liouville equation or equivalent "quantum BBGKY hierarchy", - for e.g. standard electron-phonon Hamiltonians, - also is actually important problem [19].

At today's stage of development of statistical mechanics it is useful to state that unprejudiced treatment of this science inevitably discovers flicker fluctuations of rates of transport processes, even diffusivity (rate of random walk) of particle in ideal gas [15, 16, 20, 22] and, moreover, even in the formal Boltzmann-Grad limit (under vanishingly small gas parameter) [30].

This fact excellently highlights inconsistency of attempts to reduce 1/f-noise and related long-living statistical correlations and dependences to some very long memory or relaxation times. And thus it highlights inconsistency of the underlying opinion that any statistical correlations between random phenomena gives up some literal or at least indirect physical correlations between them.

In the next Section by means of elementary logics only we shall show that in reality in many-particle systems just physical disconnectedness of inter-particle collisions leads to uncertainty and 1/f-noise of relative frequency of collisions and rate of wandering of each particle. Thus we from a new viewpoint shall justify both the general logics of Introduction and the following elementary mathematical analysis of molecular random walk.

IV. MYTHS AND REALITY OF RANDOM WALKS

A. Gaussian probability law and two meanings of independence of random events

First, recall why the Gaussian law have appeared and appears in various theoretical models. This is because it naturally comes from assumption of statistical independence of BP' displacements (increments of random walk) at non-intersecting time intervals. And, most importantly, because physicists have gotten accustomed to identify statistical independence of random events in the sense of the probability theory with their independence in the sense of their non-influencing one on another.

Both these circumstances have more than three hundred years history. A history of the Gaussian law had taken beginning from the celebrated "law of large numbers" [35] discovered by J. Bernoulli who investigated statistics of sequences of observations on vicissitudes of life or, for instance, coin tossing or playing dice, under assumption that unpredictable outcomes of successive "random trials" are mutually independent. To be more precise, that their probabilities are independent, that is joint probability of several random events decomposes (factorizes) into product of their individual probabilities.

Exactly in such the way the (statistical) independence is introduced in modern probability theory [33]. But there it is nothing but formal mathematical definition, and therefore, - as A. Kolmogorov warned in [33], - deduction of this probability property from seeming independence of physical phenomena as such is possible only as a hypothesis to be verified by experiments.

In other words, any evidences of independence of physical random events at every concrete their realization, - in the sense, for instance, of absence of cause-and-consequence connections between them, - as such can not be sufficient ground for declaring statistical independence of these events in a set (statistical ensemble) of realizations (observations).

Logically inverting this thesis, we obtain that even when statistical experiments reveal statistical dependence in an ensemble of realizations of random events, this observation does not necessarily mean existence of some real interaction of the events. Just such situations do occur when one meets 1/f-noise.

Hence, identifying of the two meanings of "independence" is nothing but fallacy. Unfortunately, it tradition-

ally governs relations of physicists to randomness, even despite its careful disclosure, - from viewpoint of fundamental statistical mechanics, - by N. Krylov more than sixty years ago [36].

B. Collisions, chaos and noise in system of hard balls

Mathematicians know N. Krylov as one of pioneers of modern theory of dynamical chaos. According to it [37, 38], for instance, motion of $N \geq 3$ elastic hard balls in a box or disks on torus, obeying deterministic laws of mechanics, is indistinguishable from a random process [38, 39]. For us here, it is important that statistical characteristics of this process are crucially sensible to ratio of its observation time and total number of balls participating in it.

More precisely, let us consider role of parameter $t/\tau N$, where τ is mean free path time of a given ball and thus characteristic time of relaxation of its velocity because of collisions with other balls [14, 16]. At

$$t/\tau \gg N\,,$$

clearly, number $\propto t/\tau$ of quantities describing trajectory of any particular ball is much greater than number $\propto N$ of quantities establishing initial state of the whole system, so that each particular trajectory contains one and the same exhaustive information on the system. Moreover, this information is contained even in any small part of the particular trajectory with duration $\sim N\tau \ll t$.

Due to this circumstance, fluctuations in numbers of collisions of given ball from any time sub-interval $\sim N\tau$ to next one behave like statistically independent random values, or “white noise” (which is well understandable: presence of some relationship or correlation between them would be recognition of some system’s initial state specificity yet non-realized on shorter time intervals, in contradiction to the condition $t \gg N\tau$). Correspondingly, relative frequency of the ball’s collisions time-averaged over whole observation time is almost nonrandom, that is one and the same for all balls and all initial conditions (at fixed full system’s energy, of course), while statistics of fluctuations in number and rate of collisions (of given ball) at intervals $t \gg N\tau$ obeys the law of large numbers, i.e. is asymptotically Gaussian.

C. Paradox of independence

Such the picture of chaos of collisions like usual noise is quite pleasant for physicists. But we should not forget that it had required the condition $t/\tau \gg N$ establishing rigid (detetrministic) non-local in time and space (nonvanishing at $t/\tau \to \infty$ and spanning all the balls) physical (cause-and-consequence) inter-dependence between collisions. Just at the expense of this dependence, - paradoxically! - the statistical independence of time-distant and space-distant events (collisions) was ensured.

In other words, interestingly, creation of ideal disorder, - with which statistical independence is usually associated, - needs vigilant underlying control of it and, in this sense, global strict order. At this point we involuntarily remind how Dront in the B. Zakhoder’s Russian translation of the L. Carroll’s “Alice’s adventures in wonderland” agitated other personages to “fit into strict disorder”, or “stand up strictly anyhow”. Along with these laughable words, comparisons suggest themselves with the mysterious “quantum non-locality” and “entangled quantum states”.

D. Uncertainty and 1/f-noise of relative frequency of collisions and rate of diffusion

However, in the real world it is not simple to mark off temporal disorder of random events in so strict way as to subordinate it to the law of large numbers. It is not simple by those simple reason that real many-particle systems are characterized by just opposite ratio of duration of observations (practically achievable in experiments) and number of particles in the system:

$$t/\tau \ll N\,.$$

Therefore, the appeal to arbitrary large averaging times, so much beloved in mathematical physics, has no factual grounds [39].

The above inverse inequality is satisfied even for rather small volumes of solids and fluids isolated from the rest of the world [16]. All the more, this inequality is true if one takes into account physical impossibility of complete isolation and hence necessity to include to N particles (and generally degrees of freedom) of all huge surroundings of a system under interest. And definitely this inequality covers objects of the Gibbs statistical mechanics, in which number of particles N is not limited, and which was under N. Krylov’s critical analysis [36].

Now, number $\propto t/\tau$ of quantities sufficient for description of observed trajectory of one or another particle (ball) all the time stays small as compared with number $\propto N$ of independent causes, i.e. variables of system‘s (initial) state, determining the trajectory.

But averaging over relatively few number of consequences determined by much larger number of causes definitely is unable to produce a certain result, since the result remains dependent on many unknown free parameters and does not represent all possible variants of course of events, all the more can not represent them under some certain proportion. Therefore, time averaging of observations of particle’s motion in any particular experiment (at each realization of system’s phase trajectory) inevitably brings unpredictably new value of relative frequency of the particle’s collisions, all the more, new distribution

(histogram) of collisions (or more complex events) in respect to their inner characteristics. In other words, an experimenter meets 1/f-noise (see Introduction).

From here we see that, instead of fabrication of hypotheses on relative frequencies or "probabilities" and "independences" of events constituting random walks it would be better for all that to follow Newton [8] and devote ourselves to investigation of equations of (statistical) mechanics.

E. Game of independences and problems of statistical mechanics

Just said is just to what N, Krylov called in his book [36] clarifying falseness of the widespread prejudices (citation [42]) "... as if a probability law exists regardless of theoretical scheme and full experiment" and "... as if "obviously independent" phenomena should have independent probability distributions".

The "full experiment" here means concrete realization of system's phase trajectory considered as a single whole, - as an origin of practical observations, - without its artificial division into "independent" time fragments (thus, we in Sections 2 and 3 above have analyzed just a full experiment).

As far as, - at $N \gg t/\tau$, - time-smoothed relative frequency, or rate, of a given sort of random phenomena or events (collisions of a given particle with others) varies from one experiment to another, demonstrating non-self-averaging, we can not (have no grounds to) introduce for such event a separately definite (individual) a priori "probability". This means that all the events occur seeming commonly statistically dependent, since, figuratively speaking, all equally are responsible for resulting, each time new, rate of their appearance (a posteriori probability). This is so in spite of that physically all the events are independent, since at $N \gg t/\tau$ are determined by interactions with different groups from total set of N particles. Consequently, we come to crash (inapplicability) of the Bernoulli's law of large numbers based on postulate of statistical independence.

Here, we clearly see another side of "paradox of independence": a true full-value chaos implies infinitely long statistical dependences and correlations.

It is clear also why the molecular Brownian motion, being conjugated with such full-valued chaos, does not want to go into "Procrustean bed" of Gaussian statistics and, all the more, Boltzmann's kinetics.

V. CONCLUSION

Unfortunately, the above underlined popular careless ideas of independences and probabilities of random phenomena (once again citing [36]) "... are so much habitual that even a person who had agreed with our argumentation then usually automatically returns to them as soon as he faces with a new question. The origin of stableness of these ideas is in that they are based on common intuitive notion about statistical laws, and therefore they would be permissible and advisable if the talk concerned learning of phenomena of empirical reality. However, such ideas turn out to be quite unsatisfactory as a bench-mark for substantiation of probability laws when the talk is about connections between statistical laws to principles of the micro-mechanics".

Fortunately, at present we have understanding of errors of replacing micro-mechanics by speculative probabilistic constructions, let beautiful in themselves and likely. Besides, as we noted above, there is already an experience of consecutive investigation of equations of statistical mechanics in application to transport processes. It clearly shows that mechanics of systems of very many interacting particles, or degrees of freedom, in now way prescribes for the interactions to keep definite rates of changing system's micro-state (transition probabilities), even when molecular chaos takes form of a macroscopic order (let even thermodynamic equilibrium).

The point is that any realization of "elementary" act of interactions in fact is a product of full (initial) micro-state of the system, so that number of causes of visible randomness always highly exceeds number of its manifestations under time averaging even in most long realistic experiments. As the consequence, any particular experiment presents to researcher's eyes its own unique assortment of relative frequencies ("probabilities"), or time rates, of random events composing a process under observations. That is just the 1/f-noise.

Hence, being surprised at 1/f-noise is not more reasonable than being surprised at noise in general. The Nature needs 1/f-noise as expression of all inexhaustible resources of the Nature's randomness in any particular "irreversible" processes as well as in originality of the wholly observed realization of our Universe's evolution at all its time scales. A purely stochastic world, without 1/f-noise, in which anything can be easily time-averaged, would be too tedious (and even, possibly, would repress a free will [40]).

Unfortunately, as we have seen above, 1/f-noise involves a "bad" statistics absolutely alien to the law of large numbers and resembling one what sometimes enforces its observers, - for example, in [41], - to suspect action of mysterious "cosmic factors". This fact significantly complicates theoretical tasks.

Fortunately, although an influence from cosmos never is undoubtedly excluded, a source of randomness quite sufficient for 1/f-noise creation is contained, - as we noted above, - already in so simple system as molecular Brownian particle interacting with ideal gas. And, generally, - as we have demonstrated above, - a source of 1/f-noise definitely exists in any medium which allows Brownian motion. Hence, one has every prospect of success in building and experimental verification of theory of 1/f-noise and accompanying statistical anomalies starting from very usual Hamiltonians.

We believe that the presented notes will induce somebody of interested readers to work in this intriguing area of statistical physics.

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